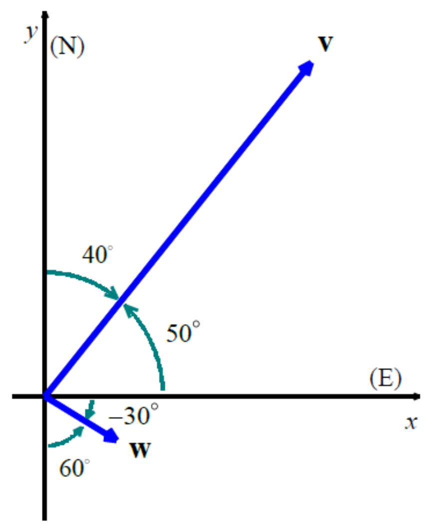
9.2 The Unit Vector and Vector Applications Notes

Using Vectors in Component Form to Solve Applications

We continue our discussion of component forms of vectors from **Section 9.1** and resume the process of **resolving** vectors into their components. This next example revisits **Example 9.1.2**, making use of component forms and vector algebra to solve this problem.

Example 9.2.1. A plane leaves an airport with an airspeed of 175 miles per hour at a bearing of N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

Solution. We proceed as we did in **Example 9.1.2** and let $\mathbf{v}$ denote the plane's velocity and $\mathbf{w}$ denote the wind's velocity, and set about determining $\mathbf{v}+\mathbf{w}$. If we regard the airport as being at the origin, the positive y-axis as acting as due north and the positive x-axis acting as due east, we see that the vectors $\mathbf{v}$ and $\mathbf{w}$ are in standard position and their directions correspond to the angles 50° and -30°, respectively.



Hence, the component forms:

$$\mathbf{v} = 175\left\langle \cos(50^\circ), \sin(50^\circ)\right\rangle = \left\langle 175\cos(50^\circ), 175\sin(50^\circ)\right\rangle$$

$$\mathbf{w} = 35\left\langle \cos(-30^\circ), \sin(-30^\circ)\right\rangle = \left\langle 35\cos(-30^\circ), 35\sin(-30^\circ)\right\rangle$$

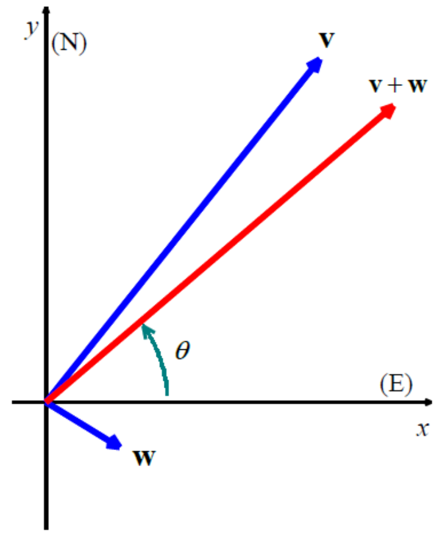
Since we have no convenient way to express the exact values of cosine and sine of 50°, we leave both vectors in terms of cosines and sines. Adding corresponding components, we find the resultant vector:

$$\mathbf{v}+\mathbf{w} = \left\langle 175\cos(50^\circ)+35\cos(-30^\circ), 175\sin(50^\circ)+35\sin(-30^\circ)\right\rangle$$

To find the true speed of the plane, we compute the magnitude of the resultant vector:

$$\begin{aligned}\|\mathbf{v}+\mathbf{w}\| &= \sqrt{\left(175\cos(50^\circ)+35\cos(-30^\circ)\right)^2+\left(175\sin(50^\circ)+35\sin(-30^\circ)\right)^2} \\ &\approx 184\end{aligned}$$

Hence, the true speed of the plane is approximately 184 miles per hour. To find the true bearing, we need to find the angle θ which corresponds to the polar form (r,θ), $r>0$, of the point $(x,y)=\left(175\cos(50^\circ)+35\cos(-30^\circ),175\sin(50^\circ)+35\sin(-30^\circ)\right)$. Since both of these coordinates are positive[1], we know θ is a Quadrant I angle, as depicted below.



Furthermore,

$$\begin{aligned}\tan(\theta) &= \frac{y}{x}\\ &= \frac{175\sin(50^\circ)+35\sin(-30^\circ)}{175\cos(50^\circ)+35\cos(-30^\circ)}\end{aligned}$$

Using the arctangent function, we get $\theta \approx 39^\circ$. Since, for the purposes of bearing, we need the angle between $\mathbf{v}+\mathbf{w}$ and the positive y-axis, we take the complement of θ and find the true bearing of the plane to be approximately N51°E.

The Unit Vector

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**.

Any nonzero vector divided by its magnitude is a unit vector. If $\mathbf{v}$ is a nonzero vector, then $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a 'unit vector in the **direction** of $\mathbf{v}$'. Noting that magnitude is always a scalar, and that dividing by a scalar is the same as multiplying by its reciprocal, a unit vector for any nonzero vector $\mathbf{v}$ can be found through multiplication by $\frac{1}{\|\mathbf{v}\|}$. The process of multiplying a nonzero vector by the reciprocal of its magnitude is called '**normalizing** the vector'. We leave it as an exercise to show that $\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}$ is a unit vector for any nonzero vector $\mathbf{v}$.

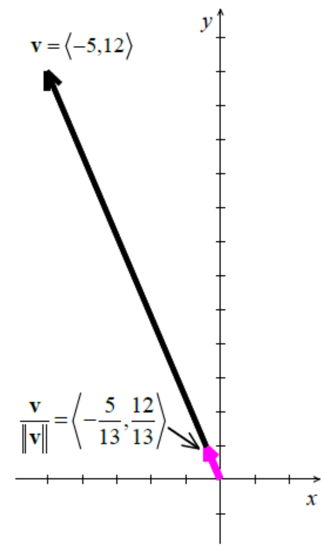
Example 9.2.2. Find a unit vector in the same direction as $\mathbf{v} = \langle -5, 12 \rangle$.

Solution. We begin by finding the magnitude.

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Next, we divide $\mathbf{v} = \langle -5, 12 \rangle$ by $\|\mathbf{v}\| = 13$.

$$\begin{aligned} \frac{\mathbf{v}}{\|\mathbf{v}\|} &= \left(\frac{1}{\|\mathbf{v}\|} \right) \mathbf{v} \\ &= \frac{1}{13} \langle -5, 12 \rangle \\ &= \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle \end{aligned}$$



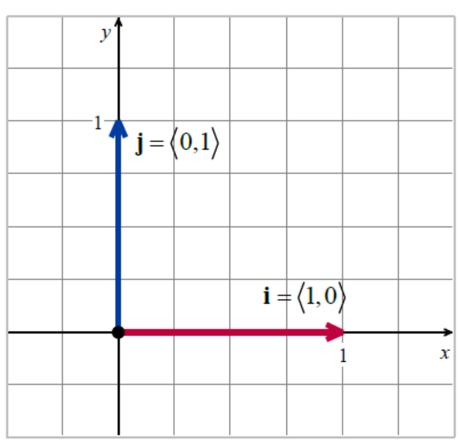
We can check that $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$ is indeed a unit vector by verifying that its magnitude is 1. Try it!

The Principal Unit Vectors

Of all of the unit vectors, two deserve special mention.

Definition. The Principal Unit Vectors:

- The vector $\mathbf{i}$ is defined by $\mathbf{i} = \langle 1, 0 \rangle$.
- The vector $\mathbf{j}$ is defined by $\mathbf{j} = \langle 0, 1 \rangle$.



We can think of the vector $\mathbf{i}$ as representing the positive x-direction while $\mathbf{j}$ represents the positive y-direction. We have the following 'decomposition' theorem.[3]

Theorem 9.4. Principal Vector Decomposition Theorem:

Let $\mathbf{v}$ be a vector with component form $\mathbf{v} = \langle v_1, v_2 \rangle$. Then $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$.

The proof of **Theorem 9.4** is straightforward. Since $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$, we have from the definition of scalar multiplication and vector addition that

$$\begin{aligned}
v_1\mathbf{i} + v_2\mathbf{j} &= v_1\langle 1,0\rangle + v_2\langle 0,1\rangle && \textbf{definition of i and j} \\
&= \langle v_1, 0\rangle + \langle 0, v_2\rangle && \textbf{scalar multiplication} \\
&= \langle v_1, v_2\rangle && \textbf{vector addition} \\
&= \mathbf{v}
\end{aligned}$$

Example 9.2.3. Given a vector $\mathbf{v}$ with initial point $P(2,-6)$ and terminal point $Q(-6,6)$, write the vector in terms of $\mathbf{i}$ and $\mathbf{j}$.

Solution.

$$\begin{aligned}
\mathbf{v} &= (-6-2)\mathbf{i} + (6-(-6))\mathbf{j} \\
&= -8\mathbf{i} + 12\mathbf{j}
\end{aligned}$$

□

Performing Operations on Vectors in Terms of i and j

When vectors are written in terms of $\mathbf{i}$ and $\mathbf{j}$, we carry out addition, subtraction and scalar multiplication by performing operations on corresponding components.

Operations on Vectors Written in terms of $\mathbf{i}$ and $\mathbf{j}$: Given $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}$, then

- $\mathbf{v} + \mathbf{w} = (v_1 + w_1)\mathbf{i} + (v_2 + w_2)\mathbf{j}$
- $\mathbf{v} - \mathbf{w} = (v_1 - w_1)\mathbf{i} + (v_2 - w_2)\mathbf{j}$
- $k\mathbf{v} = (kv_1)\mathbf{i} + (kv_2)\mathbf{j}$ for any scalar k

These results can be verified using definitions of addition, subtraction and scalar multiplication from **Section 9.1** along with **Theorem 9.4**. Their verification is left to the student.

Example 9.2.4. Use vectors $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j}$ and $\mathbf{w} = -3\mathbf{i} + \mathbf{j}$ to find $3\mathbf{v} + \mathbf{w}$.

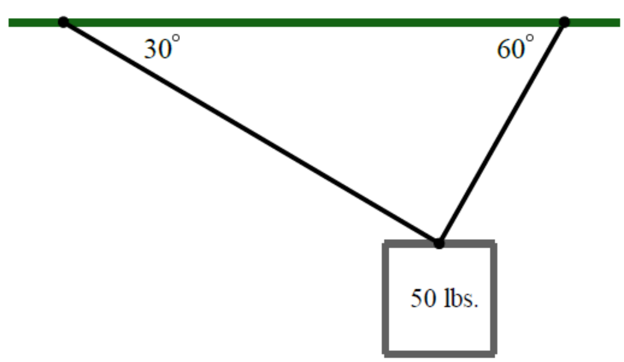
Solution.

$$\begin{aligned} 3\mathbf{v} + \mathbf{w} &= 3(4\mathbf{i} - 2\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \\ &= 3(4\mathbf{i} + (-2)\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \\ &= (12\mathbf{i} + (-6)\mathbf{j}) + (-3\mathbf{i} + \mathbf{j}) \quad \textbf{scalar multiplication} \\ &= (12 + (-3))\mathbf{i} + (-6 + 1)\mathbf{j} \quad \textbf{vector addition} \\ &= 9\mathbf{i} - 5\mathbf{j} \end{aligned}$$

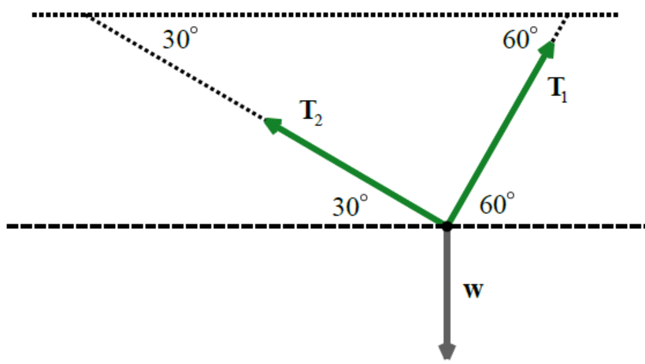
□

Example 9.2.5. A 50 pound speaker is suspended from the ceiling by two support braces. If one of them makes a 60° angle with the ceiling and the other makes a 30° angle with the ceiling, what are the tensions on each of the supports?

Solution. We first represent the problem schematically.



We have three forces acting on the speaker: the weight of the speaker, which we'll call $\mathbf{w}$, pulling the speaker directly downward, and the forces on the support rods, which we'll call $\mathbf{T}_1$ and $\mathbf{T}_2$ (for 'tensions') acting upward at angles 60° and 30°, respectively. We provide the corresponding vector diagram below.



Note that we have used alternate interior angles to determine the added angle measures in the above diagram. We are looking for the tensions on the supports, which are the magnitudes $\|\mathbf{T}_1\|$ and $\|\mathbf{T}_2\|$. In order for the speaker to remain stationary[4], we require $\mathbf{w}+\mathbf{T}_1+\mathbf{T}_2=\mathbf{0}$. Viewing the common initial point of these vectors as the origin and the dashed line as the x-axis, we find component representations for the three vectors involved.

- We can model the weight of the speaker as a vector pointing directly downward with a magnitude of 50 pounds. That is, $\|\mathbf{w}\|=50$. Since the vector $\mathbf{w}$ is directed strictly downward, $-\mathbf{j}=\langle 0,-1\rangle$ is a unit vector in the direction of $\mathbf{w}$. Hence,

$$\begin{aligned}\mathbf{w} &= 50\langle 0,-1\rangle \\ &= \langle 0,-50\rangle\end{aligned}$$

- For the force in the first support, applying **Theorem 9.3**, we get

$$\begin{aligned}\mathbf{T}_1 &= \|\mathbf{T}_1\|\langle \cos(60^\circ),\sin(60^\circ)\rangle \\ &= \|\mathbf{T}_1\|\left\langle \frac{1}{2},\frac{\sqrt{3}}{2}\right\rangle \\ &= \left\langle \frac{\|\mathbf{T}_1\|}{2},\frac{\|\mathbf{T}_1\|\sqrt{3}}{2}\right\rangle\end{aligned}$$

- For the second support, we note that the angle 30° is measured from the negative x-axis, so the angle needed to write $\mathbf{T}_2$ in component form is 150°. Hence,

$$\begin{aligned}\mathbf{T}_2 &= \|\mathbf{T}_2\|\left\langle \cos(150^\circ), \sin(150^\circ)\right\rangle \\ &= \|\mathbf{T}_2\|\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle \\ &= \left\langle -\frac{\|\mathbf{T}_2\|\sqrt{3}}{2}, \frac{\|\mathbf{T}_2\|}{2}\right\rangle\end{aligned}$$

The requirement $\mathbf{w} + \mathbf{T}_1 + \mathbf{T}_2 = \mathbf{0}$ gives us

$$\langle 0, -50\rangle + \left\langle \frac{\|\mathbf{T}_1\|}{2}, \frac{\|\mathbf{T}_1\|\sqrt{3}}{2}\right\rangle + \left\langle -\frac{\|\mathbf{T}_2\|\sqrt{3}}{2}, \frac{\|\mathbf{T}_2\|}{2}\right\rangle = \langle 0, 0\rangle$$

$$\left\langle \frac{\|\mathbf{T}_1\|}{2} - \frac{\|\mathbf{T}_2\|\sqrt{3}}{2}, \frac{\|\mathbf{T}_1\|\sqrt{3}}{2} + \frac{\|\mathbf{T}_2\|}{2} - 50\right\rangle = \langle 0, 0\rangle$$

Equating the corresponding components of the vectors on each side, we get a system of linear equations in the variables $\|\mathbf{T}_1\|$ and $\|\mathbf{T}_2\|$.

$$\begin{cases} (E1) & \dfrac{\|\mathbf{T}_1\|}{2} - \dfrac{\|\mathbf{T}_2\|\sqrt{3}}{2} = 0 \\ (E2) & \dfrac{\|\mathbf{T}_1\|\sqrt{3}}{2} + \dfrac{\|\mathbf{T}_2\|}{2} - 50 = 0 \end{cases}$$

From (E1) we get $\|\mathbf{T}_1\| = \|\mathbf{T}_2\|\sqrt{3}$. Substituting into (E2) gives

$$\begin{aligned}\frac{\left(\|\mathbf{T}_2\|\sqrt{3}\right)\sqrt{3}}{2} + \frac{\|\mathbf{T}_2\|}{2} - 50 &= 0 \\ \frac{3\|\mathbf{T}_2\| + \|\mathbf{T}_2\|}{2} &= 50 \\ \|\mathbf{T}_2\| &= 25\end{aligned}$$

Hence, $\|\mathbf{T}_2\| = 25$ pounds and $\|\mathbf{T}_1\| = \|\mathbf{T}_2\|\sqrt{3} = 25\sqrt{3}$ pounds.